# Exercise 23

Apply the Fourier transform to solve the equation

$$u_{xxxx} + u_{yy} = 0, \quad -\infty < x < \infty,\ y \geq 0,$$

satisfying the conditions

$$u(x,0) = f(x), \quad u_y(x,0) = 0 \quad \text{for } -\infty < x < \infty,$$

and $u(x,y)$ and its partial derivatives vanish as $|x| \to \infty$.

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### Solution

This exercise is the same as 1.11—$t$ has been replaced by $y$ here. The PDE is defined for $-\infty < x < \infty$, so we can apply the Fourier transform to solve it. We define the Fourier transform here as

$$\mathcal{F}\{u(x,y)\} = U(k,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u(x,y)\, dx,$$

which means the partial derivatives of $u$ with respect to $x$ and $y$ transform as follows.

$$\mathcal{F}\left\{\frac{\partial^n u}{\partial x^n}\right\} = (ik)^n U(k,y)$$
$$\mathcal{F}\left\{\frac{\partial^n u}{\partial y^n}\right\} = \frac{d^n U}{dy^n}$$

Take the Fourier transform of both sides of the PDE.

$$\mathcal{F}\{u_{xxxx} + u_{yy}\} = \mathcal{F}\{0\}$$

The Fourier transform is a linear operator.

$$\mathcal{F}\{u_{xxxx}\} + \mathcal{F}\{u_{yy}\} = 0$$

Transform the derivatives with the relations above.

$$(ik)^4 U + \frac{d^2U}{dy^2} = 0$$

Expand the coefficient of $U$ and bring the term to the right side.

$$\frac{d^2U}{dy^2} = -k^4 U \tag{1}$$

The PDE has thus been reduced to an ODE. Before we solve it, we have to transform the initial conditions as well. Taking the Fourier transform of the initial conditions gives

$$u(x,0) = f(x) \quad \rightarrow \quad \mathcal{F}\{u(x,0)\} = \mathcal{F}\{f(x)\}$$
$$U(k,0) = F(k) \tag{2}$$
$$\frac{\partial u}{\partial y}(x,0) = 0 \quad \rightarrow \quad \mathcal{F}\left\{\frac{\partial u}{\partial y}(x,0)\right\} = \mathcal{F}\{0\}$$
$$\frac{dU}{dy}(k,0) = 0. \tag{3}$$

Equation (1) is an ODE in $y$, so $k$ is treated as a constant. The solution to the ODE is given in terms of sine and cosine.

$$U(k,y) = A(k)\cos k^2 y + B(k)\sin k^2 y$$

Apply the first initial condition, equation (2).

$$U(k,0) = A(k) = F(k)$$

In order to apply the second initial condition, differentiate $U(k,y)$ with respect to $y$.

$$\frac{dU}{dy}(k,y) = -k^2 A(k)\sin k^2 y + k^2 B(k)\cos k^2 y$$

Now apply equation (3).

$$\frac{dU}{dy}(k,0) = k^2 B(k) = 0 \quad \to \quad B(k) = 0$$

Therefore, the solution to the ODE that satisfies the initial conditions is

$$U(k,y) = F(k)\cos k^2 y.$$

In order to change back to $u(x,y)$, we have to take the inverse Fourier transform of $U(k,y)$.

$$\begin{aligned} u(x,y) &= \mathcal{F}^{-1}\{U(k,y)\} \\ &= \mathcal{F}^{-1}\{F(k)\cos k^2 y\} \end{aligned}$$

Because we are taking the inverse Fourier transform of a product of two functions, $F(k)$ and $\cos k^2 y$, we can apply the convolution theorem, which states that

$$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi.$$

Looking up the inverse Fourier transform of $\cos k^2 y$ in a table,

$$\mathcal{F}^{-1}\{\cos k^2 y\} = \frac{1}{\sqrt{2y}}\cos\left(\frac{x^2}{4y} - \frac{\pi}{4}\right),$$

we can write $u(x,y)$ by the convolution theorem as

$$u(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)\frac{1}{\sqrt{2y}}\cos\left(\frac{\xi^2}{4y} - \frac{\pi}{4}\right)d\xi.$$

Pull the constant in front of the integral to obtain the final result.

$$u(x,y) = \frac{1}{\sqrt{4\pi y}}\int_{-\infty}^{\infty} f(x-\xi)\cos\left(\frac{\xi^2}{4y} - \frac{\pi}{4}\right)d\xi$$